ASYMPTOTIC BEHAVIOR OF ORDINARY
DIFFERENTIAL EQUATIONS*

M. Pinto**¹

¹Depto. de Matemáticas, Universidad de Chile, Casilla 653
Santiago-Chile

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In [1] we have obtained results over asymptotic integration for systems of the type

(1) \[ y' = f(t, y) + g(t, y), \]

where the unperturbed systems

(2) \[ x' = f(t, x) \]

and their corresponding variational systems

(3) \[ z' = f_x(t, x(t), x(0))z \]

are h-systems. If f is linear in x, i.e., f(t, x) = A(t)x then (2) and (3) coincide with the system

(4) \[ x' = A(t)x, \quad A \in C([a, \infty)). \]

This system is an h-system if there exists a constant c ≥ 1 and a continuous and positive function h defined on \( I_a = [a, \infty) \) such that

\[
|\varnothing(t)\varnothing^{-1}(s)| \leq c h(t)h(s)^{-1}, \quad t \geq s \geq a
\]

for a (then all) fundamental matrix \( \varnothing \) of (4). This condition can be understood as “h dominates the growth of the solutions of (4)". But, it is possible that only some solutions are dominated by h and the other solutions are dominated by other function k, that is there exists a projection P such that

\[
|\varnothing(t)P\varnothing^{-1}(s)| \leq c h(t)h(s)^{-1} \\
|\varnothing(t)(I-P)\varnothing^{-1}(s)| \leq c k(t)k(s)^{-1},
\]

Thus, if P = I we have, as a particular case, that (4) is an h-system.

The purpose of this paper is to extend the results for linear system (4) given in [1], to systems which possesses a dichotomic property.

Let h and k be two continuous and positive functions defined on \( I_{a_0} \).

Let \( \varnothing \) be a fundamental matrix of (4). The linear system (4) has an h-k dichotomy if there exist a projection P and a positive constant c such that

(5) \[
|\varnothing(t)P\varnothing^{-1}(s)| \leq c h(t)h(s)^{-1}, t \geq s \geq a \\
|\varnothing(t)(I-P)\varnothing^{-1}(s)| \leq c k(t)^{-1}k(s), s \geq t \geq a
\]

**Example.** A fundamental matrix for the system

\[
x' = \begin{bmatrix} -t^{-1} & 0 \\ 0 & 1 \end{bmatrix} x
\]

is given by

\[
\varnothing(t) = \begin{bmatrix} t^{-1} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}.
\]

Let P be the following matrix

\[
P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

we have

\[
|\varnothing(t)P\varnothing^{-1}(s)| = \begin{bmatrix} s & 0 \\ t & 0 \\ 0 & 0 \end{bmatrix} \leq h(t)h(s)^{-1}, t \geq s \geq a > 0
\]
and \[
\| \mathcal{O}(t)(I-P)\mathcal{O}^{-1}(s) \| = \begin{bmatrix}
0 & 0 \\
0 & e^{(s-t)}
\end{bmatrix} \leq k(s)k(t)^{-1}, \ s \geq t \geq a > 0,
\]
where \( h(t) = t^{-1} \) and \( k(t) = e^t \).

**Theorem 1**

Let \( \mathcal{O}(t) \) be the fundamental matrix of (4) with \( \mathcal{O}(a) = 1 \). Assume that (4) has an h-k dichotomy with projection \( P \) such that

\[
(6) \quad \| \mathcal{O}(t)(I-P) \| \leq c_k(t)^{-1}, \text{ for } t \geq a
\]

and

\[
(7) \quad h(t)k(t)h(s)^{-1}k(s)^{-1} \leq c_1, \text{ for } t \geq s \geq a,
\]

where \( c_1 \) is a positive constant. Let \( g = g(t,y) \) be a continuous function defines on \( I_a \times IR^n \) such that \( h(t)^{-1}g(t,0), k(t)g(t,0) \in L_1(I_a) \)

\[
(8) \quad \| g(t,y_1) - g(t,y_2) \| \leq \lambda(t) \| y_1 - y_2 \|
\]

for all \( t,y_1,y_2 \), where \( \lambda \in L_1(I_a) \).

Denote by \( V_+ = \text{Rang } P, \) by \( V_- = \text{Rang } (I-P) \) and by \( x_+ \) (respectively \( x_- \)) a solution of (4) such that \( x_+(a) \in V_+ \) (respectively \( x_-(a) \in V_- \)).

Then for any solution \( x_+ \) (respectively \( x_- \)) of (4), there exists a unique solution \( y_+ \) (respectively \( y_- \)) of the unperturbed system

\[
(9) \quad y' = A(t)y + g(t,y)
\]

such that for \( t \to \infty \)

\[
(10) \quad y_\pm = x_\pm + \| \mathcal{O}(t)P \| 0(1) + h_\pm 0(1)
\]

or shortly

\[
(11) \quad y_\pm = x_\pm + h_\pm 0(1),
\]

where \( h_+ = h, \ h_- = k^{-1} \) and \( 0(1) \) denotes a function which has a limit as \( t \to \infty \).

**Proof.** First, we no ice that if \( x_0 \in \text{Rang } P, \) the solution of (4) such that \( x(a) = x_0 \) satisfy, by (5),

\[
\| x(t) \| \leq c_1 \| x_0 \| h(t) \text{ for } t \geq a.
\]

Similarly, by (6), \( \| x(t) \| \leq c_1 \| x_0 \| k(t)^{-1} \) if \( x(a) = x_0 \in \text{Rang } (I-P) \). Now, for solution \( x \) of (4), the solutions of the integral equation

\[
y(t) = x(t) + \int_0^t \mathcal{O}(t)P\mathcal{O}^{-1}(s)g(s,y(s))ds - \int_t^\infty \mathcal{O}(t)(I-P)\mathcal{O}^{-1}(s)g(s,y(s))ds
\]

are solutions of (9). Consider the integral operators \( T_+ \) and \( T_- \) given by

\[
T_+y(t) = x_+(t) + \int_0^t \mathcal{O}(t)P\mathcal{O}^{-1}(s)g(s,y(s))ds - \int_t^\infty \mathcal{O}(t)(I-P)\mathcal{O}^{-1}(s)g(s,y(s))ds
\]
where \( |x_\pm(t)| \leq c |x_\pm| h_\pm(t) \) for \( t \geq t_0 \), \((h_+ = h, h_- = k^{-1})\).

Let be

\[
C_\pm = \{ y \in C([t_0, \infty), IR^n) | |h_\pm^{-1} y|_\infty < \infty \}
\]

\(C_+\) and \(C_-\) are Banach vectorial spaces with

\[
|y|_+ = |h_\pm^{-1} y|_\infty \quad \text{and} \quad |y|_- = |h_\pm^{-1} y|_\infty
\]

respectively. Moreover, \(T_{\pm}: C_\pm \to C_\pm\) and, by using (5), (6), (7) and (8),

\[
|T_{\pm}(y_1) - T_{\pm}(y_2)|_\pm \leq \alpha(t_0) |y_1 - y_2|_\pm
\]

where

\[
\alpha = \alpha(t_0) = c(1+c_1^2) \int_{t_0}^{\infty} \lambda(s) ds.
\]

Therefore, if \( t_0 \) is chosen so large that

\[
\alpha = c(1+c_1^2) \int_{t_0}^{\infty} \lambda(s) ds < 1,
\]

\(T_{\pm}\) has a unique fixed point \(y_{\pm}\). Since \(y_{\pm}\) verifies

\[
\left| \int_{t_0}^{t} \varnothing(t) P \varnothing^{-1}(s) g(s, y_{\pm}(s)) ds \right| \leq |\varnothing(t) P 0(1) + h_\pm(t) \cdot o(1)
\]

and

\[
\left| \int_{t_0}^{t} \varnothing(t)(I-P) \varnothing^{-1}(s) g(s, y_{\pm}(s)) ds \right| \leq h_\pm(t) \cdot o(1), \text{ for } t \to \infty
\]

it also satisfies (10). Finally, (12) implies

\[
|T_{\pm}(y_{\pm}(t)) - x_{\pm}(t)| \leq \alpha h_\pm(t) |y|_\pm
\]

and then (11) follows ending the proof.

The formula (10), or (11), generalize the formula

\[
y = x + h \cdot \tilde{o}(1)
\]

obtained in [1] when (4) is an \( h \)-system i.e. if \( P = I \).

**Remark 1.** The correspondances \( x_+ \leftrightarrow y_+ \) and \( x_- \leftrightarrow y_- \) are bicontinuous respect to be norms \( \| \|_\pm \) because

\[
(1-\alpha) \| y_2 - y_1 \|_\pm \leq \| x_2 - x_1 \|_\pm \leq (1+\alpha) \| y_2 - y_1 \|_\pm.
\]

Coppel ([2], p.77) establishes this correspondance only between bounded solutions. Their result follows by taking \( h = k = 1 \). The condition (6) is only needed to assure that \( |x(t)| \leq c |x(a)| k(t)^{-1} \) if \( x(a) \in \text{ Rang } P \).

**Remark 2.** Rang \( P \) represents the “stable manifold” and Rang \((I-P)\) the “unstable manifold”. In fact, in Theorem 3 next we will prove a version of the theorem of the stable manifold for “non-autonomous system”.
Corollary 1
Assume that (4) has an h-k dichotomy. Then
\[ y^* = (A(t) + B(t))y, \quad B \in L_1(I_\alpha) \]
has a solution matrix Y such that as \( t \to \infty \)
\[ Y = \mathcal{O}[I + \mathcal{O}^{-1} H \cdot \mathcal{o}(1)], \]
where \( H = hI_r \oplus k^{-1}I_{n-r} \), with \( I_r, I_{n-r} \) identity matrices.

Corollary 2
Suppose that (4) has an h_l-h_l^{-1} dichotomy with a projection \( P \) such that
\[ |\mathcal{O}(t)(I-P)| \leq c h_l(t) \]
and
\[ \lim_{t \to \infty} h_l(t)^{-1} |\mathcal{O}(t)P| = 0 \]
Then, if \( x_i \) is a solution of (4) such that \( |x_i(t)| \leq c h_i(t) \) for \( t \geq a \), then there exists a solutions of (13) satisfying
\[ Y_i = x_i + h_i \cdot o(1) \]
In particular, if \( \mathcal{O}^{-1}H \) is bounded and if the above condition holds for any \( i, 1 \leq i \leq n \), then (13) has a fundamental matrix Y such that
\[ Y = \mathcal{O}[I + o(1)] \]
where \( H = \text{diag } \{ h_1, ..., h_l, ..., h_n \} \)
Corollary 2 represents a linear extension of Levinson’s theorem on asymptotic integration [3] by taking
\[ h_l(t) = \exp \int_1^t \text{Re}(\lambda_i(\tau))d\tau \]
and asking to \( \left| \lambda_i(t) \right|_{i=1}^n \) to have the dichotomic conditions of the Levinson’s theorem. Thus Theorem 1 represents non-diagonal and a nonlinear extension of that result.
Moreover, Levinson’s theorem does not permit us to study directly
\[ y^* = (A + B(t))y, \quad B \in L_1(I_\alpha) \]
if A has multiple eigenvalues. With Theorem 1 we are able to study the asymptotic behaviour of the solution of (14) if i) \( \text{Re}\lambda \neq 0 \) for all eigenvalues \( \lambda \) of A (independent of the multiplicity) and ii) \( \text{Re}\lambda = 0 \) only for simple eigenvalues (with no restriction on the others). In fact, for i), by the Jordan canonical form, there exist a projection P and positive constants \( c, \alpha, \beta \) such that
\[ |e^{tA}P e^{-sA}| \leq c e^{-\alpha(t-s)}, \quad t \geq s \geq a \]
\[ |e^{tA}(I-P)e^{-sA}| \leq c e^{-\beta(s-t)}, \quad s \geq t \geq a \]
and
\[ |e^{tA}(I-P)| \leq c e^{\beta t}, \quad t \geq a \]
for any \( a \in \mathbb{IR} \).

Then, by Corollary 2, there exists a fundamental matrix \( Y \) of (13), such that
\[
Y(t) = e^{tA} + H \cdot \Theta(1),
\]
where \( H = e^{-\alpha t} I_n \oplus e^{\beta t} \cdot o(1)n_{r} \) and \( r \) is the number of eigenvalues \( \lambda \) with \( \Re\lambda < 0 \). For ii), the same is true with \( \beta = 0 \). Thus, by (10), we obtain the following.

**Corollary 3**

If \( A \) is a constant matrix and the solutions of
\[
x' = Ax
\]
are bounded, then there exists a fundamental matrix \( Y \) of (14) such that
\[
(15) \quad Y(t) = e^{tA} + o(1).
\]
This is a more precise result than those given in [1].

In the next precise theorem we obtain a result for conditionally integrable coefficients.

**Theorem 2**

Suppose that (4) has an \( h-k^{-1} \) dichotomy. \( V(t) \) be a continuous matrix such that
\[
Q(t) = \int_{t}^{\infty} V(s)ds
\]
exists for \( t \geq a \) and let \( g=g(t,y) \) as in Theorem 1. Suppose that \( AQ, QA, VQ \in L_{1}(I_{n}) \).

Then, the conclusion of Theorem 1 holds for the system
\[
(16) \quad y' = (A(t) + V(t))y + g(t,y)
\]

**Proof.** Using the formula
\[
(I-Q(t))^{-1} = I + (I-Q)^{-1}Q
\]
valid for \( t \) large and making
\[
y = (I-Q)z,
\]
the system (16) is transformed into the system:
\[
z' = [A-AQ-VQ + (I-Q)^{-1} Q(A+AQ+VQ)]z + (I-Q)^{-1} g(t,(I-Q)z).
\]

Now, by Theorem 1 the result follows.

Th. 2 extends Th. 3.1 [4] to a non-linear and non-diagonal version. Also it permits linear extensions of the results in [5] and [6].

In the following, we study some results for integrable dichotomies.

**Theorem 3**

Suppose that there exists a projection \( P \) and a positive constant \( c_{1} \) such that for \( t \geq a \)
\[
(17) \quad \int_{a}^{t} \left| \Theta(t)P\Theta^{-1}(s) \right| ds + \int_{t}^{\infty} \left| \Theta(t)P\Theta^{-1}(s) \right| ds \leq c_{1},
\]
where \( \Theta \) is the fundamental matrix of (4) such that \( \Theta(a)=I \).
Let \( g \) be a continuous function such that: For any \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon) > 0 \) such that for \( |y_1|, |y_2| \leq \delta \) and \( t \geq a \)

\[
(18) \quad |g(t,y_1) - g(t,y_2)| \leq \epsilon |y_1 - y_2| .
\]

Then, if \( r \) is the dimension of \( \text{Rang } P \), there exists a real \( r \)-dimensional manifold \( M \) containing the origin such that any solution \( y \) of

\[
(19) \quad y' = A(t)y + g(t,y)
\]

with \( y(a) \) on the manifold \( M \) satisfies \( y(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

Moreover, there exists a \( \delta_1 \leq \delta \) such that any solution \( y \) of (19) near the origin with initial condition \( y(a) \) not belonging to \( M \) does not satisfy \( |y(t)| \leq \delta_1, t \geq a \). If \( g \) is analytic in \( y \) for each \( t \geq a \) and \( |y| \) is small, then \( M \) is an analytic manifold.

**Proof.** By lemma 1, p. 68 in Coppel [2]

\[
(20) \quad \lim_{t \rightarrow \infty} |\mathcal{O}(t)P| = 0
\]

and there exists a positive constant \( K \) such that

\[
(21) \quad |\mathcal{O}(t)P y_o| \leq K |y_o| , \text{ for } t_o \geq a
\]

For any continuous function \( y \) defined on \( I_a \), let \( T \) be the integral operator

\[
Ty(t) = \mathcal{O}(t)Py_o + \int_a^t \mathcal{O}(t)P\mathcal{O}^{-1}(s)g(s,y(s))ds - \int_t^\infty \mathcal{O}(t)(I-P)\mathcal{O}^{-1}(s)g(s,y(s))ds,
\]

where \( y_o \) is a constant vector.

For \( ||y_1||, ||y_2|| \leq \delta \) we have that, by (17) and (18),

\[
(22) \quad ||Ty_1 - Ty_2|| \leq \epsilon c_1 ||y_1 - y_2|| ,
\]

where \( ||y|| = ||y||_\infty \).

Moreover, if \( \epsilon c_1 < 1 \) and \( |y_o| < K^{-1} (1-\epsilon c_1)r \), \( T \) transforms the ball \( ||y|| \leq r, r \leq \delta \), into itself. Then, by contraction principle, there exists a unique fixed point \( y \) of \( T \):

\[
(23) \quad y = Ty
\]

Hence \( y \) is a solution of (19) and \( y(t) \rightarrow 0 \) as \( t \rightarrow \infty \). In fact, let

\[
\ell = \lim_{t \rightarrow \infty} \sup \{ |y(t)| \}
\]

and \( \beta \) a constant such that \( \epsilon c_1 < \beta < 1 \). If \( \ell > 0 \), there exists \( t_o \geq a \) such that \( |y(t)| \leq \beta^{-1} \ell \) for \( t \geq t_o \). Then (23) implies, for \( t \geq t_o \),

\[
|y(t)| \leq |\mathcal{O}(t)P| y_o + \int_{t_o}^t \mathcal{O}^{-1}(s)g(s,y(s))ds + \epsilon c_1 \beta^{-1} \ell
\]

Hence, if \( t \rightarrow \infty \)

\[
\ell \leq \epsilon c_1 \beta^{-1} \ell
\]

which is impossible. Thus \( \ell = 0 \).
Now

(24) \(Py(a) = Py_o\)

and

(25) \((I-P)y(a) = \int_a^\infty (I-P)\Theta^{-1}(s)g(s,y(s))ds = \tilde{g}(Py_o)\)

By (24), we obtain \(r\) components \((r = \text{dim Rang } P)\) of \(y(a)\), say \(|y_{\sigma_i}|^r\). The other \(n-r\) components \(|y_{\sigma_i}|_{i=r+1}^n\) satisfy (25), i.e.

\(y_{\sigma_j} = \tilde{g}_j(y_{\sigma_1}, y_{\sigma_2}, ..., y_{\sigma_n}), \quad j = r+1, ..., n.\)

These equations define a \(r\) manifold \(M\). We will show that a solution \(y\) of (19) such that \(y(a)\) is small enough and belonging to \(M\) satisfies \(y(t) \rightarrow 0\) as \(t \rightarrow \infty\). Suppose \(|y(t)| \leq \delta\) for \(t \geq a\). Since the integral in (25) converges, the solution \(y\) verifies

(26) \(y(t) = \Theta(t)Py(t_a) + \Theta(t)(I-P)v + \int_a^t \Theta(t)P\Theta^{-1}(s)g(s,y(s))ds\)

\[ -\int_t^\infty \Theta(t)(I-P)\Theta^{-1}(s)g(s,y(s))ds, \]

where

\(v = y(a) + \int_a^\infty (I-P)\Theta^{-1}(s)g(s,y(s))ds\)

Hence (26) implies \(\Theta(t)(I-P)v\) must be bounded as \(t \rightarrow \infty\), i.e. \((I-P)v = 0\) because \(\Theta(t)(I-P)\) is not bounded by Lemma 2, p. 74 Coppel [2]. Therefore \(y(a)\) is on \(M\), \(y\) satisfies \(y = Ty\) and by uniqueness \(y(t) \rightarrow 0\) as \(t \rightarrow \infty\).

Finally if \(g\) is analytic in \(y\) for \(y\) small and \(t \geq a\) the fixed point \(y = y(t,y_o)\) of \(T\) is also analytic as function of \(y_o\) for \(t\) fixed hence \(\tilde{g}\) is so and \(M\) is an analytic manifold.

With respect to asymptotic integration, if the solution \(y\) of (19) satisfies that \(y(a)\) is point in \(M\) small enough then there exists a unique \(x\) solution of (4) such that

\(y = x + o(1)\)

for \(t \rightarrow \infty\). This is the relation between the bounded solutions of (4) and (19). Is there a similar relation for unbounded solutions?

The differentiability of \(g\), as the analitycity, is also herited by the manifold \(M\).

**Theorem 4**

The manifold \(M\) of Theorem 3 is \(C_k\) if \(g\) is so.

**Proof.** Let \(y(t,y_o)\) be the solution of (22) and denote by

\(d(t) = |y(t,y_o + he) - y(t,y_o)|,\)

where \(e_i = (0, ..., 1, ..., 0)\) and \(h \in \mathbb{R}\). By using (23), (17), (18) and (21), we have for \(h\) and \(|y_o|\) small that

\(d(t) \leq K|h| + |e|d||\).
hence $||d|| \leq (1-\varepsilon c_1)^{-1}K \cdot h$. Thus, if we denote
\[ \psi_h(t,y_o) = h^1d(t) \]
it follows $|\psi_h(t,y_o)| \leq (1-\varepsilon c_1)^{-1}K = K_1$. From (23), we have
\[
(27) \quad \psi_h(t,y_o) = \mathcal{O}(t)Pv_1 + \int_a^t \mathcal{O}(t)P\mathcal{O}^{-1}(s)\left[ g_y(s,y(s,y_o))\psi_h(s,y_o) + r_h(s)\right]ds \\
- \int_t^\infty \mathcal{O}(t)(I-P)\mathcal{O}^{-1}(s)\left[ g_y(s,y(s,y_o))\psi_h(s,y_o) + r_h(s)\right]ds,
\]
where $g_y$ is the Jacobian matrix and $r_h(s) = h^{-1}\{g(s,y(s,y_o + he)) - g(s,y(s,y_o))\} - g_y(s,y(s,y_o))\psi_h(s,y_o)$

By using the theorem of the mean and the continuity of $g_y$ for small $s$ and by (18) and (20) for large $s$, it follows that there exist $\eta(h)$ such that $\eta(h) \to O$ as $h \to O$ and
\[ |\psi_h(s,y_o)| \leq \eta(h) \]
Consider next the “limit” integral equation
\[
(28) \quad z(t,y_o) = \mathcal{O}(t)Pv_1 + \int_a^t \mathcal{O}(t)P\mathcal{O}^{-1}(s)g_y(s,y(s,y_o))z(s,y_o)ds \\
- \int_t^\infty \mathcal{O}(t)(I-P)\mathcal{O}^{-1}(s)g_y(s,y(s,y_o))z(s,y_o)ds
\]
which possesses a solution $z$ for $|y_o|$ small because, from (18) and (20),
\[ |g_y(s,y(s,y_o))| \leq n \varepsilon. \]
Moreover, $\psi_h \to z$ as $h \to O$. In fact, subtracting (28) from (27), it follows
\[ ||z - \psi_h|| \leq \varepsilon c_1 ||z - \psi_h|| + c_1K_1\eta(h)\]
Thus, choosing $\varepsilon c_1, < 1, \psi_h \to z$ as $h \to O$, i.e.,
\[ \partial y(t,y_0)/\partial y_o \]
exists and is the solution $z$ of (28).

REFERENCES


